Counting holomorphic cylinders via non-archimedean geometry

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This talk is about enumerative geometry

1. Counting curves (Kontsevich's recursive formula)
2. Counting discs (Wall-crossing formula)
3. Tools from non-archimedean geometry and tropical geometry
4. Counting cylinders in log Calabi-Yau surfaces

Counting curves

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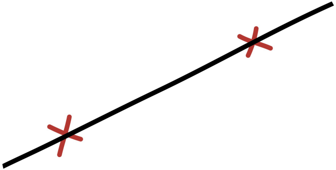

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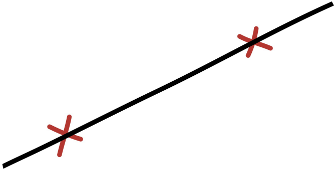

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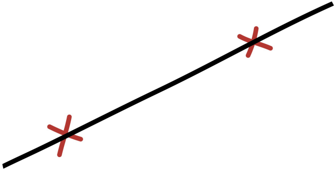

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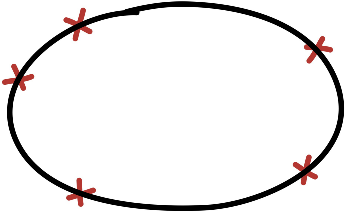

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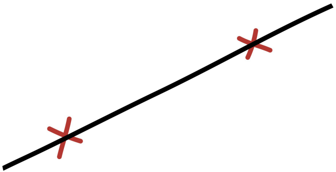

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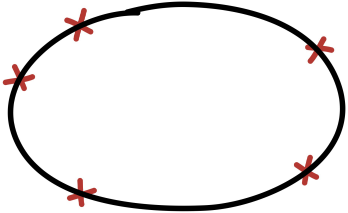

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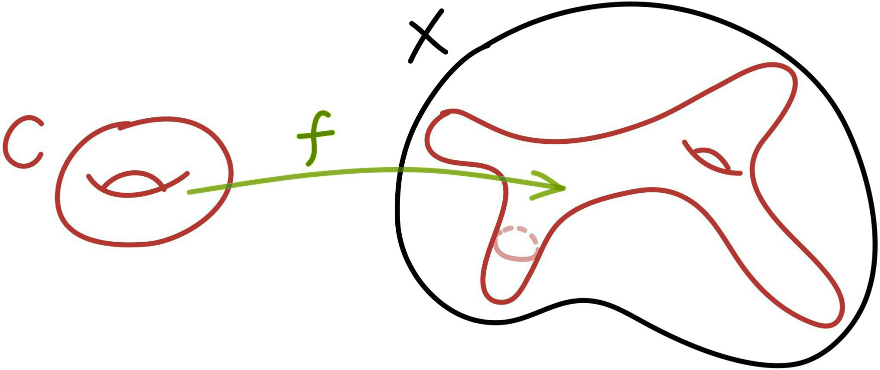

Explanations:

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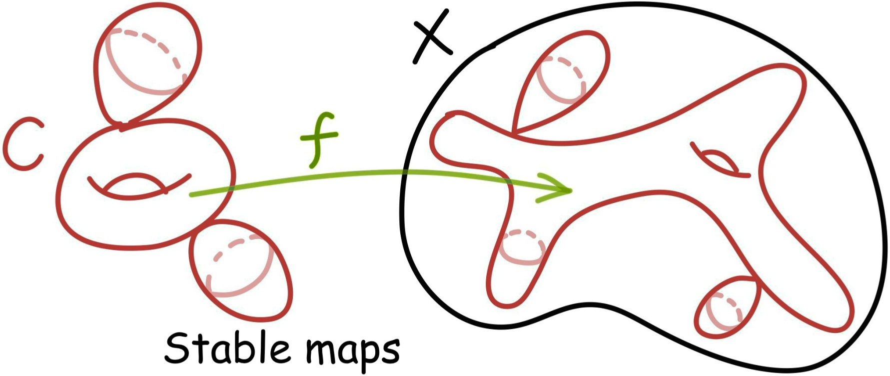

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Example: $X = \mathbb{C}P^2$

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Proof.

$$\begin{aligned} f : \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^2 \\ (u, v) &\longmapsto (P(u,v), Q(u,v), R(u,v)) \end{aligned}$$

where P, Q, R are homogeneous polynomials of degree d.
Now it is easy to count the dimension. □

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Let N_d denote the number of degree d rational curves in $\mathbb{C}P^2$ passing through $3d-1$ generic points.

We have $N_1 = 1$, $N_2 = 1$, $N_3 = 12$, $N_4 = 620$ (Zeuthen 1874), $N_5 = ?$, $N_6 = ?, \dots$

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Theorem (Kontsevich-Manin 94, Ruan-Tian 94)

The numbers N_d satisfy the following recursive formula

$$N_d = \sum_{\substack{d_1,d_2>0\\ d_1+d_2=d}} \binom{3d-4}{3d_1-2} (d_1 d_2)^2 N_{d_1} N_{d_2} - \sum_{\substack{d_1,d_2>0\\ d_1+d_2=d}} \binom{3d-4}{3d_2-1} d_1 d_2^3 N_{d_1} N_{d_2}.$$

Now we can compute $d_5 = 87304$, $d_6 = 26312976$, $d_7 = 14616808192$, $d_8 = 13525751027392$, $d_9 = 19385778269260800, \dots$

Gromov-Witten invariants

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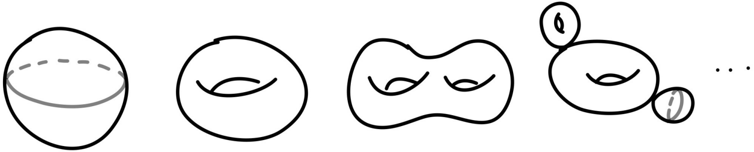
Remark

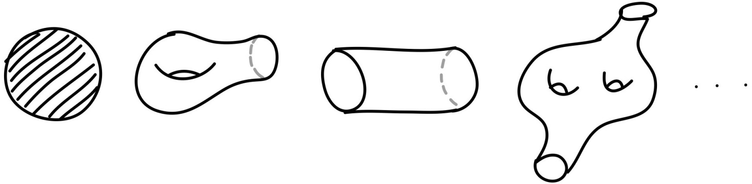
- *The numbers N_d are examples of Gromov-Witten invariants.*
- *The recurrence relation above is a particular case of WDVV equations for Gromov-Witten invariants.*

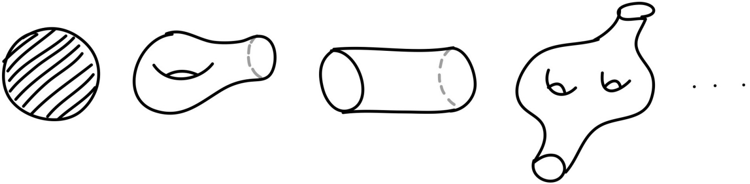
Counting discs

Inspired by string theory and mirror symmetry, one tries to count not only "closed curves" in a target space X



but also "open curves" (i.e. Riemann surfaces with boundaries)





A lot more possibilities!

In order to obtain a finite number of them, we need not only fix some marked points, but also impose boundary conditions. Let me explain through an example.

Let $X = Y \setminus D$, where Y is a complex projective surface, $D = D_1 + \cdots + D_l \in |-K_Y|$ is an anti-canonical cycle of rational curves.

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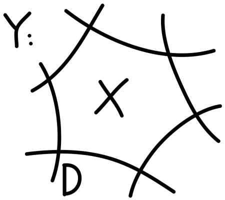

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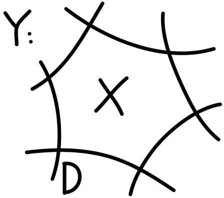
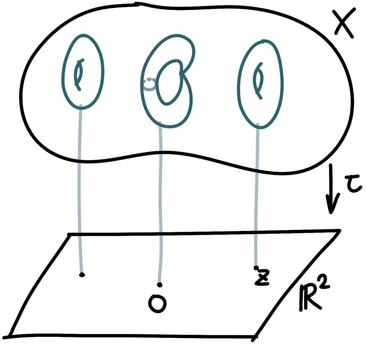
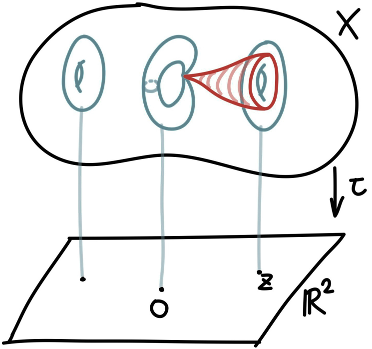




We have a projection $\tau\colon X \to \mathbb{R}^2$, such that the fibers outside the origin are real 2-dimensional tori.

For a point $z \in \mathbb{R}^2 \setminus O$, a class $\gamma \in H_1(\tau^{-1}(z), \mathbb{Z})$, let $\Omega_z(\gamma)$ be the number of holomorphic discs whose boundary sits in the fiber $\tau^{-1}(z)$ and represents the class γ.

Q: What properties do the numbers $\Omega_z(\gamma)$ have?



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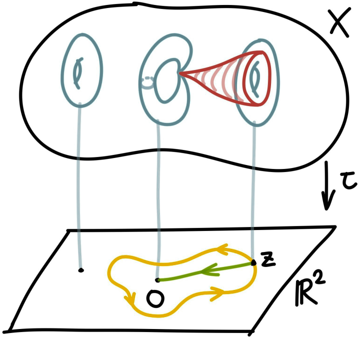
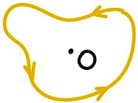
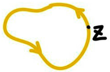


Q: What properties do the numbers $\Omega_z(\gamma)$ have?

A: Let γ_z be the primitive vector in the direction $\overrightarrow{zO}$. We define a generating series $a_z(t) = \sum_{k\geq 1} k\Omega_z(k\gamma_z)t^k$, and an automorphism of $\mathbb{C}[\![u_1, u_2]\!]$ given by $T_z\colon \mathbf{u}^\beta \mapsto a_z(\mathbf{u}^{\gamma_z})^{\langle\beta,\gamma_z\rangle} \cdot \mathbf{u}^\beta, \beta \in \mathbb{Z}$.

Theorem (A reformulation of Gross-Pandharipande-Siebert 09)

$\forall$ loop  in $\mathbb{R}^2 \setminus O$, we have $\prod T_z = \mathrm{id}$ (product over z along the loop )

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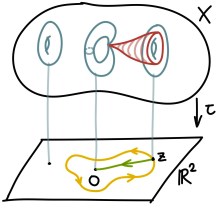

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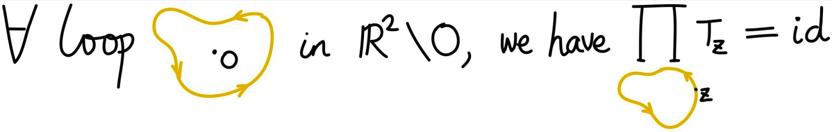

Remark

- *This is a particular case of the Kontsevich-Soibelman wall-crossing formula for Donaldson-Thomas invariants.*
- *Like the recursive formula for the numbers N_d, the wall-crossing formula permits us to compute all the numbers $\Omega_z(\gamma)$ starting from simple initial data.*

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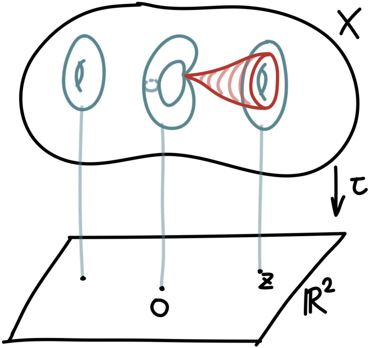

The projection $\tau\colon X \to \mathbb{R}^2$ just now is a particular case of the retractions of a non-archimedean analytic space X to its skeleton.

Q: What is non-archimedean here in the story?

Deformation retraction

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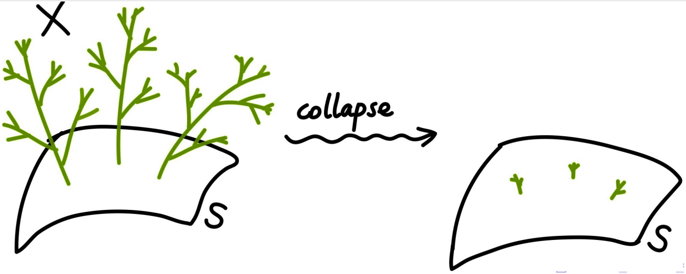
A: A family of complex algebraic varieties X_t parametrized by a punctured disc gives rise naturally to an analytic space X defined over the non-archimedean field $k = \mathbb{C}((t))$. (Think of X as the total space of the family.)

Theorem (Berkovich 99)

Given a nice formal model of X, one can construct a strong deformation retraction from X to a polyhedral complex S.



Example of K3 surface

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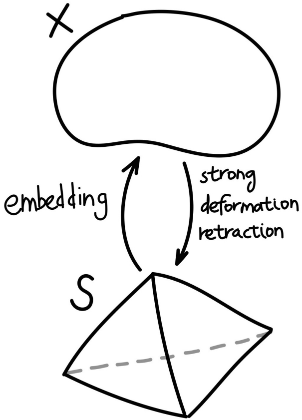
X : K3 surface of type III degeneration

The skeleton S is a polyhedral complex homeomorphic to S^2.



First steps in enumerative non-archimedean geometry

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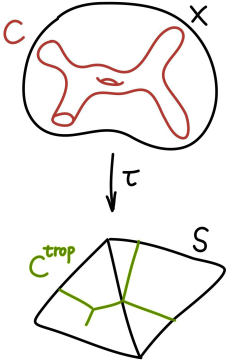
A: Here are the first steps I developed in the last few years:

Theorem (Y, arXiv:1304.2251)

Under the retraction τ, any holomorphic curve C in X becomes a piecewise linear graph C^{trop} in S which satisfies the generalized balancing conditions.



We call C^{trop} the *tropical curve* associated to C.

The balancing conditions are constraints on the shape of C^{trop} around every vertex.

They are determined by the intersection theory on the formal model.

Tropicalization of families of curves

In the theorem above, we considered the tropicalization of a single holomorphic curve.

Q: What about families of curves?

A: Fix a real number A, set-theoretically we have

$$\overline{\mathcal{M}}_{g,n}(X,A) \coloneqq \{\, n\text{-pointed genus } g \text{ stable maps into } X \text{ with area} \leq A \,\}$$

$$\Big\downarrow \tau_{\mathcal{M}}$$

$$\mathcal{M}(S,A) \coloneqq \{\, \text{tropical curves in } S \text{ with area} \leq A \,\}\,.$$

Theorem (Y, arXiv:1401.6452)

$\overline{\mathcal{M}}_{g,n}(X,A)$ is a proper k-analytic stack if X is proper.

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$\mathcal{M}(S,A)$ is a finite compact polyhedral complex. The map $\tau_{\mathcal{M}}$ is continuous. Its image is compact and polyhedral.

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Given these preparations, now the enumeration goes as follows:

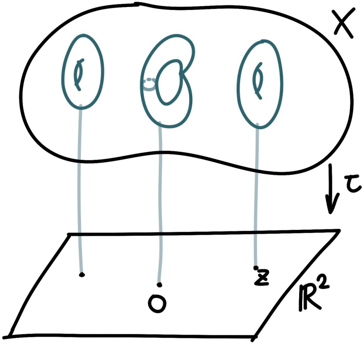
1. Put enough constraints such that all tropical curves in question become rigid.
2. Let Z be a rigid tropical curve. Using the theorems on the previous page, one proves that $\tau_{\mathcal{M}}^{-1}(\{Z\})$ is a proper k-analytic stack.
3. Construct a virtual fundamental class on $\tau_{\mathcal{M}}^{-1}(\{Z\})$, and obtain the number $N(Z)$ of holomorphic curves which tropicalizes to Z.
4. Sum of $N(Z)$ over all possible tropical curves Z.

Counting cylinders in log Calabi-Yau surfaces

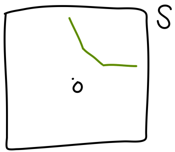
Let's consider again the situation $X = Y \setminus D$, constant family over $\mathbb{C}((t))$, where Y is a projective surface, and $D = D_1 + \cdots + D_l \in |-K_Y|$ is an anti-canonical cycle of rational curves.
We counted discs inside X ten minutes ago, now let's count cylinders.

Lemma

In this case the skeleton S is a homeomorphic to $\mathbb{R}^2$. The retraction map $\tau\colon X \to S$ is a k-analytic torus fibration outside $O \in S$.



Goal: Define a virtual number of holomorphic cylinders $N(L, \beta)$ given a class $\beta \in \mathrm{NE}(Y)$ and a broken path L in $S \setminus O$.



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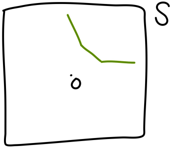
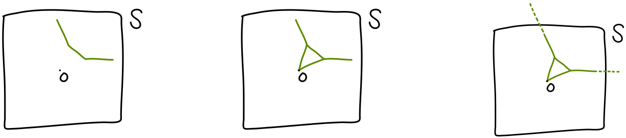


Step 1: broken path $L \rightsquigarrow$ tropical cylinder $\rightsquigarrow$ extended tropical cylinder



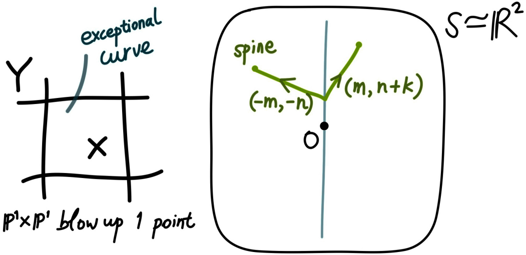
Step 2: By toric blowup, we produce two divisors $\widetilde{D}_1, \widetilde{D}_2 \subset \widetilde{Y}$ corresponding to the two unbounded edges.

Step 3: Let $\mathcal{M}(\widetilde{Y}, L, \beta)$ denote the corresponding moduli stack of holomorphic curves. We prove that it is a proper k-analytic stack. (Rigidity of the tropical cylinder + the theory of formal models of k-analytic stable maps developed in arXiv:1401.6452)

Step 4: Virtual fundamental class $\rightsquigarrow$ virtual number $N(L, \beta)$.

Example (focus-focus singularity)

Example: $Y : \mathbb{P}^1 \times \mathbb{P}^1$ blowup a smooth point in the toric boundary:



We obtain that the corresponding virtual number of cylinders equals $\binom{m}{k}$.

It gives exactly the wall-crossing formula around a focus-focus singularity:

$$(x, y) \longmapsto (x(1+y), y)$$

In particular, $x^m y^n \longmapsto x^m(1+y)^m y^n = \sum_{k=0}^{m} \binom{m}{k} x^m y^{k+n}.$