Marzinkiewicz-Zygmund sequences in real algebraic varieties

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Theorem (Marzinkiewicz-Zygmund)

Let $\Lambda_n = \{e^{\frac{2ij\pi}{n+1}}\}_{j=0}^{n}$ be the $(n+1)$-roots of unity and let $1 < p < \infty$. There are constants C_p, such that

$$\frac{C_p^{-1}}{n} \sum_{\lambda \in \Lambda_n} |q(\lambda)|^p \leq \int_0^{2\pi} |q(e^{it})|^p \, dt \leq \frac{C_p}{n} \sum_{\lambda \in \Lambda_n} |q(\lambda)|^p$$

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It is interesting to know what other possible sequences Λ_n are possible.

Beurling-Landau necessary conditions

An asymptotic density

Assume now that $p = 2$ for simplicity, and that Λ_n are a collection of points with the same property as in the statement of the theorem. Then

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If Λ_n are a sequence of finite sets with the Marzinkiewicz-Zygmund property for $q = 2$, then for any subinterval $I \subset \mathbb{T}$ we have

$$\liminf_{n\to\infty} \frac{\#(\Lambda_n \cap I)}{n} \geq \frac{|I|}{|\mathbb{T}|}$$

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The normalized Lebesgue measure in $\mathbb{T}$ is the critical Nyquist density in this context.

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We endow $\mathcal{P}_k$ with the norm given by $L^2(\mu)$. We assume that μ is not degenerate, i.e. μ is not supported on the the zero set of a polynomial $p \not\equiv 0$, otherwise we should work with a subvariety of M.

Marzinkiewicz-Zygmund sequences

Let $\Lambda = \{\Lambda_k\}_k \subset M$ be a sequence of finite sets of points of M.

Definition

We say that Λ is a Marzinkiewicz-Zygmund sequence if there is a constant $C > 0$ such that

$$C^{-1} \sum_{\lambda \in \Lambda_k} \frac{|p(\lambda)|^2}{c_{\lambda,k}} \leq \int_M |p|^2 \, d\mu \leq C \sum_{\lambda \in \Lambda_k} \frac{|p(\lambda)|^2}{c_{\lambda,k}}, \qquad \forall p \in \mathcal{P}_k,$$

with a natural normalization $c_{\lambda,k}$.

We are interested in the geometric distribution of points in Λ.

The normalization

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This can be computed as follows. Take $p_1, \ldots, p_{N_k}$ an orthonormal basis of $\mathcal{P}_k$ and construct:

$$K_k(z, w) = \sum_j p_j(z) p_j(w),$$

then $c_{\lambda,k} = K_k(\lambda, \lambda)$. Moreover K_k is the reproducing kernel:

$$p(z) = \int_M K_k(z, w) p(w)\, d\mu(w), \qquad \forall p \in \mathcal{P}_k$$

Frames and M-Z sequences

The orthogonal projection $L^2(\mu) \to \mathcal{P}_k$ is given by the integral kernel K_k. We denote by κ_λ the normalized reproducing kernel $\kappa_\lambda(z) = K_k(\lambda, z)/\sqrt{K_k(\lambda, \lambda)}$.

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Then Λ is a M-Z sequence if and only if the normalized reproducing kernels form a frame in $\mathcal{P}_k$, i.e.:

$$C^{-1} \int_M |p|^2 \, d\mu \le \sum_{\lambda \in \Lambda_k} |\langle p, \kappa_\lambda \rangle|^2 \le C \int_M |p|^2 \, d\mu \quad \forall p \in \mathcal{P}_k$$

Two main examples

So far this is a rather abstract problem. We consider two more concrete situations where it is more explicit:

- The variety $M = \mathbb{R}^n$. Let Ω be an open bounded convex set in M and we take $\mu = \chi_\Omega dx$.

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In this two settings we can prove

- If $x \in \Omega$, then $K_k(x,x) \simeq \min(k^{n+1}, \frac{k^n}{\sqrt{d(x,\partial\Omega)}})$.

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- If $x \in \Omega$, then $K_k(x,x) \simeq \min(k^{n+1}, \frac{k^n}{\sqrt{d(x,\partial\Omega)}})$.
- In the case of smooth manifolds: $K_k(x,x) \simeq k^n \simeq N_k$.

Landau's strategy

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- The local dimension in a domain U can be interpreted as $\int_U K_k(z,z)d\mu$.
- Altogether the above points show that if Λ is M-Z then asymptotically Λ_k has a density bigger than some critical Nyquist density.

The Nyquist density

We try to identify which is the critical density. We can use the following result:

Theorem (Berman, Boucksom, Witt-Nyström)

If μ is a Bernstein-Markov measure then

$$K_k(x,x)d\mu(x) \xrightarrow{*} \mu^{eq}.$$

The Bernstein-Markov condition is technical and it is satisfied by our both main examples. The measure μ^{eq} is the equilibrium measure.

The equilibrium potential

The variety M can be naturally embedded as the real points of a complex variety X. Given a compact $K \subset M$ and any $z \in X$ one defines the Siciak-Zaharjuta equilibrium potential as

$$u_K(z) = \sup\{\frac{1}{\deg(p)} \log |p(z)| : \sup_K |p| \leq 1.\}$$

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Then the equilibrium measure is defined as the Monge-Ampere of u_K

$$\mu^{eq} = (i\partial\bar{\partial}u_K)^n.$$

The equilibrium measure is a positive measure supported on K.

What does μ^{eq} look like?

The measure μ^{eq} is a well-known object in pluripotential theory. In the examples we mentioned before it is well understood.

- If Ω is an open bounded convex set in M and $\mu = \chi_\Omega dx$ then

$$d\mu^{eq}(z) \simeq d(z, \partial\Omega)^{-1/2} dz.$$

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$$d\mu^{eq}(z) \simeq d(z, \partial\Omega)^{-1/2} dz.$$
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- If M is a compact smooth manifold and $d\mu$ is the Lebesgue measure then $\mu^{eq} \simeq \mu$.

Main result

Theorem

If Λ is a Marzinkiewicz-Zygmund sequence in a real algebraic affine variety M endowed with a non-degenerate measure then

$$\liminf_{k\to\infty} \frac{1}{N_k} \sum_{\lambda\in\Lambda_k} \delta_\lambda \geq \mu_{eq}.$$

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In particular for the two main examples, given any ball B in the support of μ we have

$$\liminf_{k\to\infty} \frac{\#(\Lambda_k \cap B)}{N_k} \geq \frac{\mu^{eq}(B)}{\mu^{eq}(M)},$$

thus μ^{eq} is the Nyquist density.

Bernstein inequality

One of the ingredients of the proof is a Bernstein type inequality of independent interest:

Theorem

Given $M \subset \mathbb{R}^m$ be a smooth compact manifold. TFAE:

- *There is $C > 0$ such that for all polynomials p:*

$$\int_M |\nabla_t p| dV_M \leq C \deg p \int_M |p| dV_M.$$

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- *There is a uniformly separated Λ such that*

$$\int_M |p|^2 dV_M \lesssim \frac{1}{k^n} \sum_{\lambda \in \Lambda_k} |p(\lambda)|^2 \lesssim \int_M |p|^2 dV_M, \ \forall p \in \mathcal{P}_k(\mathbb{R}^m).$$

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- *M is algebraic.*

The Kantorovich-Wasserstein distance

Given a compact metric space K we defines the K-W distance between two probability measures μ and ν supported in K as

$$KW(\mu, \nu) = \inf_{\rho} \iint_{K \times K} d(x, y) d\rho(x, y),$$

where ρ is an admissible probability measure, i.e. the marginals of ρ are μ and ν respectively.

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$$KW(\mu, \nu) = \inf_{\rho} \iint_{K \times K} d(x, y) d|\rho|(x, y),$$

where ρ is an admissible complex measure, i.e. the marginals of ρ are μ and ν respectively

The complex transport plan

The K-W distance metrizes the weak-$*$ convergence. We want to prove that $KW(b_k, \sigma_k) \to 0$, where $b_k = K_k(x,x)d\mu(x)$ is the Bergman measure and σ_k is a measure such that $\sigma_k \leq \frac{1}{N_k} \sum_{\lambda \in \Lambda_k} \delta_\lambda$. Since we know already that $b_k \overset{*}{\rightharpoonup} \mu^{eq}$ that will prove the result.

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The transport plan ρ_k that is convenient to estimate is:

$$\rho_k(x,y) = \frac{1}{N_k}\sum_{\lambda\in\Lambda_k}\delta_\lambda(y)\times g_\lambda(x)\frac{K_k(\lambda,x)}{\sqrt{K_k(\lambda,\lambda)}}\,d\mu(x),$$

where g_λ is the dual frame to the normalized reproducing kernels $\{\frac{K_k(\lambda,x)}{\sqrt{K_k(\lambda,\lambda)}}\}_{\lambda\in\Lambda_k}$

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The two marginals of ρ_k are

- $\nu_k := \frac{1}{N_k} K_k(x,x)\, d\mu(x) \overset{*}{\rightharpoonup} \mu^{eq}$

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and

$$KW(\nu_k, \sigma_k) \leq \frac{1}{N_k} \sum_{\lambda \in \Lambda_k} \int d(\lambda, x) |g_\lambda(x)| \frac{|K_k(\lambda, x)|}{\sqrt{K_k(\lambda, \lambda)}}\, d\mu(x).$$

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Thus

$$KW^2(\nu_k, \sigma_k) \lesssim \frac{1}{N_k} \iint d^2(x,y) |K_k(x,y)|^2\, d\mu(x)\, d\mu(y).$$

An off-diagonal estimate

Given a bounded function f on M we denote by T_f be the Toeplitz operator on $H_k(M) \cap L^2(\mu)$ with symbol f, i.e. $T_f := \Pi_k \circ f\cdot$ where Π_k denotes the orthogonal projection from $L^2(M, \mu)$ to $H_k(M)$.

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$$\mathrm{Tr}\, T_f^2 - \mathrm{Tr}\, T_{f^2} = \frac{1}{2} \int_{M \times M} (f(x) - f(y))^2 \, |K_k(x, y)|^2 \, d\mu(x) \otimes d\mu(y)$$

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Now, setting $f := x_i$ we observe than on H_{k-1}, $T_f(p) = x_i p$. Therefore $T_{f^2} - T_f^2 = 0$ on H_{k-2}. Therefore:

$$\mathrm{Tr}\, T_f^2 - \mathrm{Tr}\, T_{f^2} = O(k^{n-1})$$

and

$$KW^2(\nu_k, \sigma_k) \lesssim \frac{1}{k}.$$