Symplectic and Poisson structures with symmetries in interaction

Eva Miranda

UPC-Barcelona

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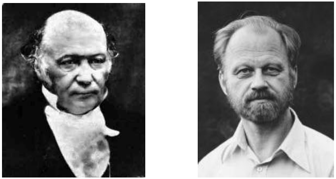
Outline

Symplectic structures

- A symplectic structure is a non-degenerate closed 2-form ω.
- Non-degeneracy gives a natural isomorphism between $T^*(M)$ and $T(M)$.
- For every f, there is a unique vector field X_f (Hamiltonian vector field), $\iota_{X_f}\omega = -df$



$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

Figure: Sir William Rowan Hamilton, Jürgen Moser and Hamilton's equations.

Jürgen Moser classified symplectic structures on surfaces using the path method associated to this formula. In this case, a symplectic form is just an *area form*.

Symplectic structures in dimension 2

Theorem (Moser)

Two symplectic structures ω_0 and ω_1 on a compact symplectic surface with $[\omega_0] = [\omega_1]$ are symplectically equivalent.

Idea behind: Moser's path method

The linear path $\omega_t = (1-t)\omega_0 + t\omega_1$ is a path of symplectic structures $\rightsquigarrow$ (Moser's trick)

$$\iota_{X_t}\omega_t = -\alpha$$

with $\omega_1 - \omega_0 = d\alpha \rightsquigarrow$ integration of the flow of X_t given by the path method gives the diffeomorphism.

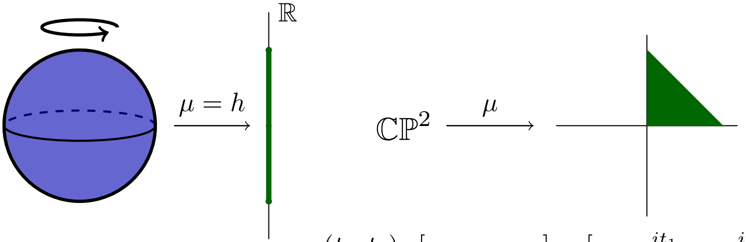
Higher dimensions?

Some classification schemes are possible with additional data (toric manifolds).

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

Moment maps in Symplectic Geometry

Definition (Symplectic case)

Let G be a compact Lie group acting symplectically on (M, ω).
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$-d\mu^X = \iota_{X^\#}\omega, \tag{1}$$

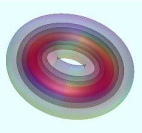
with $\mu^X = < \mu, X >$.
The map μ is called the **moment map**.

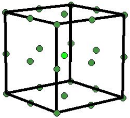
Why toric?

Given an integrable system $F = (f_1, \dots, f_n)$ ($\{f_i, f_j\} = 0$) and a compact fiber,

Theorem (Arnold-Liouville)

There exist semilocal action-angle coordinates $(p_1, \theta_1, \dots, p_n, \theta_n)$ such that $\omega = \sum_{i=0}^{n} dp_i \wedge d\theta_i$, $F = (p_1, \dots, p_n)$ with linear Hamiltonian flow on the torus.





Liouville tori (left) and Bohr-Sommerfeld orbits read from the polytope (right)

Some applications:

- Singular fibrations: Symplectic Morse-Bott classification (M.-Zung).
- Geometric Quantization: Guillemin-Sternberg, Sniaticky, Hamilton, M., Solha.

Symplectic surfaces with singularities (Radko's surfaces)

Given an oriented surface S (compact or not) with a distinguished union of curves Z, we want to modify the volume form on S by making it "explode" when we get close to Z. We want this "blow up" process to be controlled.

Figure: A Radko surface

What does "controlled" mean here? We want that the 2-form looks locally $\omega = \frac{1}{y} dx \wedge dy$ (for points in Z).

Dimension 2

(Radko) The invariants of b-symplectic structures in dimension 2 are :

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $\lim_{\epsilon\to 0} V_h^{\epsilon}(\Pi) = \int_{|h|>\epsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$.

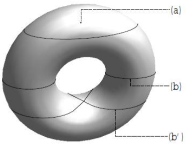
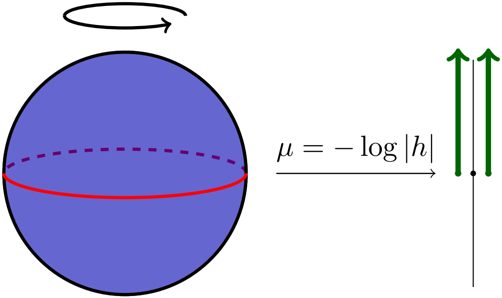


Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') are not admissible.

Radko surfaces and their symmetries

$$(S^2, \frac{1}{h} dh \wedge d\theta) \leftrightsquigarrow (S^2, h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}).$$

We want to study generalizations of rotations on a sphere.



Surfaces and circle actions

Surfaces and circle actions

The only orientable compact surfaces admitting an effective action by circles are the two sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2$ and the action is equivalent to the standard action by rotations.

In the symplectic case the standard rotation on $\mathbb{T}^2$ is not Hamiltonian (only symplectic).

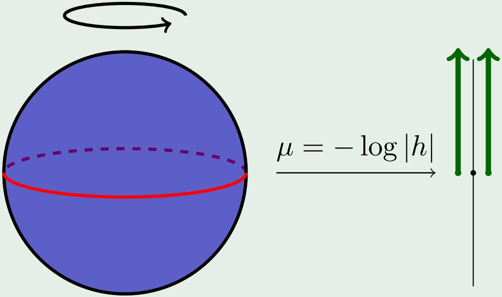
$$d\theta_1 \wedge d\theta_2(\frac{\partial}{\partial\theta_1}, \cdot) = d\theta_2.$$

In the b-symplectic case, the toric surfaces are either the sphere or the torus.

The S^1-b-sphere

Example

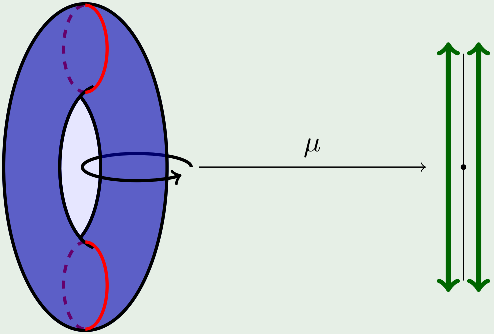
$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h, \theta) = -\log|h|$.



The S^1-b-torus

Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = -\log\left|\frac{1+\cos(\theta_1)}{\sin(\theta_1)}\right|$.



The b-line

The b-line is constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern and $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity to prescribe a smooth structure.

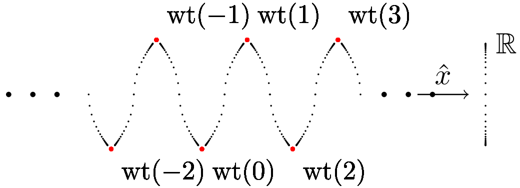
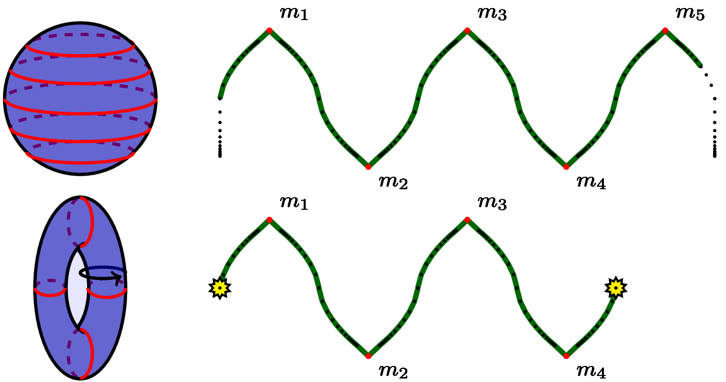


Figure: A weighted b-line with $I = \mathbb{Z}$

The weights are given by the modular periods associated to each connected component of Z.

b-surfaces and their moment map

A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



Classification of toric b-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric b-symplectic surface is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$, where Z is a collection of latitude circles.

The action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

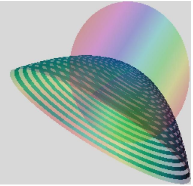
The weights $w(a)$ of the codomain of the moment map are given by de modular periods of the connected components of the critical hypersurface.

Poisson structures

A Poisson structure is a bivector field Π (i..e a section of $\Lambda^2(TM)$) with $[\Pi, \Pi] = 0$. It generalizes the notion of a symplectic structure.

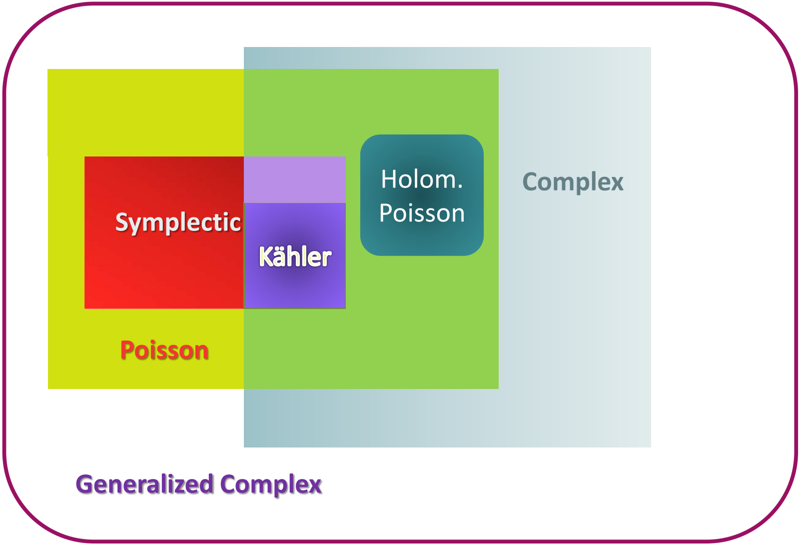
Indeed, the Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point (Weinstein's splitting theorem).

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$



This defines a symplectic foliation.

Concerned geometries...



b-Symplectic/b-Poisson/ log-symplectic

Definition

Let (M^{2n}, Π) be an oriented Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a **Poisson b-structure** on (M, Z).

Other singularities

It is possible to generalize this definition (M.-Planas-Scott) to consider more general Poisson structures.

Symplectic foliation of a Poisson b-manifold

The symplectic foliation has dense symplectic leaves and codimension 2 symplectic leaves whose union is Z.

Higher dimensions: Some compact examples.

- The product of (R, π_R) a Radko compact surface and a (S, π) a compact symplectic manifold.
- Take (N, π) be a regular corank 1 Poisson manifold and let X be a Poisson vector field. the product $S^1 \times N$ with the bivector field

$$\Pi = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi$$

is a b-Poisson manifold as long as,

1. the function f vanishes transversally.
2. The vector field X is transverse to the symplectic leaves of N.

We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

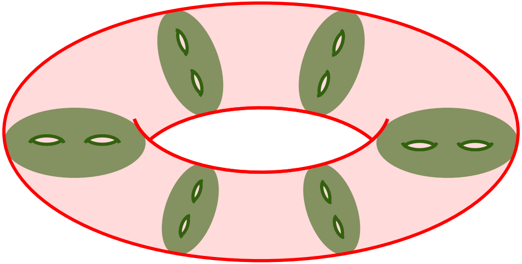
This last example is semilocally the *canonical* picture of a b-Poisson structure.

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field transverse to the symplectic foliation induced on Z.
3. Given a regular corank 1 Poisson structure, there exists a semilocal extension to a b-Poisson structure if an only if two foliated cohomology classes of the symplectic foliation vanish.

The singular hypersurface

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.



A dual approach...

- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-tangent bundle).
- A vector field v is a b-**vector field** if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

b-calculus

b-calculus was developed by Richard Melrose to formalize differential calculus on manifolds with boundary

$$\operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}).$$

In particular he obtained a proof of the Atiyah-Patodi-Singer index theorem.

A dual approach...

- The b-**cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$. The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A b-**symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.
- For b-toric actions we allow Hamiltonian functions $H_X \in {}^bC^\infty(M)$ (b-forms of degree 0).

Definition

An action of $\mathbb{T}^n$ on a b-symplectic manifold (M,ω) is a **Hamiltonian action** if:

- for each $X \in \mathfrak{t}$, the b-one-form $\iota_{X^\#}\omega$ is exact (i.e., has a primitive $H_X \in {}^bC^\infty(M)$)
- for any $X, Y \in \mathfrak{t}$, we have $\omega(X^\#, Y^\#) = 0$.

The action is **toric** if it is effective and the dimension of the torus is half the dimension of M.

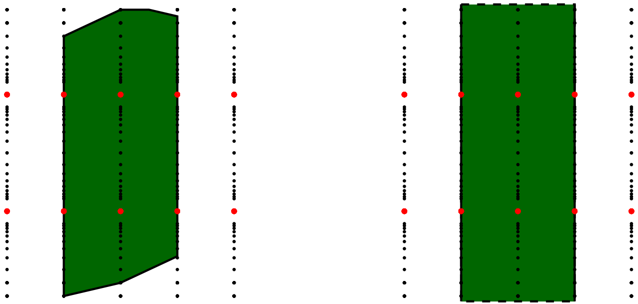
b-moment map μ such that

$$< \mu(p), X >= H_X(p),$$

but we will have to allow $\mu(p)$ to take values of $\pm\infty$, so we need to extend the pairing to accommodate that.

From local to global....

We can reconstruct the b-Delzant polytope from the Delzant polytope on a mapping torus via *symplectic cutting* in a neighbourhood of the critical hypersurface.



This information can be recovered by doing reduction by stages: Hamiltonian reduction of an action of $\mathbb{T}_Z^{n-1}$ and the classification of toric b-surfaces.

The semilocal model

Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold is

$$M_{\text{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\text{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\text{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

A b-Delzant theorem

Theorem (Guillemin,M.,Pires, Scott)

The maps that send a b-symplectic toric manifold to the image of its moment map

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*\} \tag{2}$$

and

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle)\} \to \{b\text{-}Delzant\ polytopes\ in\ {}^b\mathfrak{t}^*/\langle N\rangle\} \tag{3}$$

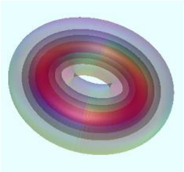
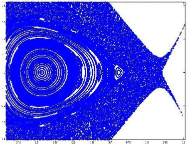
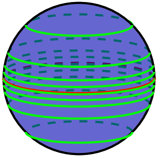
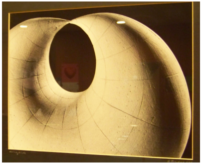
are bijections.

Toric b-manifolds can be of two types:

1. ${}^b\mathbb{T}^2 \times X$ (with X a toric symplectic manifold of dimension $(2n-2)$)
2. ${}^b\mathbb{S}^2 \times X$ and manifolds obtained via symplectic cutting (for instance, $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$).

Some applications

- A convexity theorem for $\mathbb{T}^k$-actions on b-symplectic manifolds (Guillemin- M.-Pires-Scott).
- Applications to Quantization and localization (Guillemin-M.-Weitsman).
- An action-angle theorem for integrable systems (Kiesenhofer-M.).
- A KAM theorem for b-symplectic manifolds (Kiesenhofer-M.).
- Study of semitoric systems (Kiesenhofer-M.).

Rigidity

- Symplectic manifolds: Given two close symplectic actions ρ_0 and ρ_1 on a compact symplectic manifold (M^{2n}, ω) Palais theorem + Moser's path method $\rightsquigarrow$ the actions are conjugated $\rightsquigarrow$ rigidity
- Poisson manifolds:

Theorem (M-Monnier-Zung)

Let μ and λ be two "close" moment maps (corresponding to a semisimple Lie algebra action of compact type) on a compact manifold (or neighbourhood of an invariant submanifold) then there exists a Poisson diffeomorphism such that $\mu \circ \Phi = \lambda$.

First approach

An action gives an element in $\mathcal{M} = Hom(G, Diff(M))$. Consider the additional action,

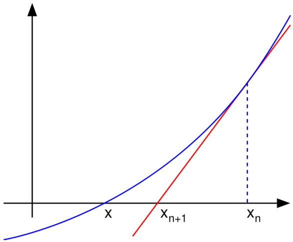
$$\begin{aligned} \beta : \quad Diff(M) \times \mathcal{M} &\longmapsto \mathcal{M} \\ (\phi, \alpha) &\mapsto \phi \circ \alpha \circ \phi^{-1} \end{aligned}$$

α_0 and α_1 are conjugated if they are on the same orbit under $\beta \rightsquigarrow$ If β has open orbits $\rightsquigarrow$ rigidity.

- The tangent space to the orbit of $\beta = 1$-coboundaries of the group cohomology with coefficients in $V = Vect(M)$; the tangent space to $\mathcal{M} =$ 1-cocycles.
- Generalized Whitehead lemma: Compact $G \rightsquigarrow H^1(G; Vect(M)) = 0 \rightsquigarrow$ tangent space to the orbit $=$ tangent space to $\mathcal{M}$.
- If $\mathcal{M}$ is a manifold (tame Fréchet) we can apply the inverse function theorem (Nash-Moser) to go from the tangent space to the manifold (tame Fréchet).

Infinitesimal rigidity implies rigidity

- In general it is hard to verify the "tame Fréchet" condition but we can apply the method used in the proof of Nash-Moser's theorem (Newton's iterative method).



- This methods allows to prove several results of type infinitesimal rigidity $\rightsquigarrow$ rigidity.
- For Hamiltonian actions on Poisson manifold we consider the Chevalley-Eilenberg complex associated to the representation given by the moment map.

Infinitesimal rigidity implies rigidity

1. Assume now that $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ are close moment maps. The difference $\phi = \mu_0 - \mu_1$ defines a 1-cochain which is a near 1-cocycle.
2. Define Φ as the time-1-map of the Hamiltonian vector field $X_{S_t(h(\phi))}$ with h the homotopy operator and S_t is a smoothing operator.
3. Define the Newton iteration,

$$\phi_d = \phi^1_{X_{S_t(h(\eta_d))}}$$

with $\eta_d = (\mu_1 - \mu_0) \circ \phi_{d-1}$. This converges to a Poisson diffeomorphism that conjugates both actions.

Applications

The general scheme proved by Monnier-M.-Zung for SCI spaces has had other applications to:

- Generalized complex geometry, Bailey-Gualtieri.
- Normal forms for submanifolds in Poisson Geometry, Crainic-Marcut.
- Poisson Lie groups Esposito-M.

References

- V. Guillemin, E. Miranda and A. Pires, *Symplectic and Poisson Geometry on b-manifolds*, Adv. Math. 264 (2014), 864–896.
- V. Guillemin, E. Miranda, A. Pires and G. Scott, *Toric actions on b-manifolds*, Int. Math. Res. Not. IMRN, 2014.
- E. Miranda, P. Monnier and N. T. Zung, *Rigidity for Hamiltonian actions on Poisson manifolds,* Adv. Math. 229 (2012), no. 2, 1136-1179.