Constructive Galois Theory

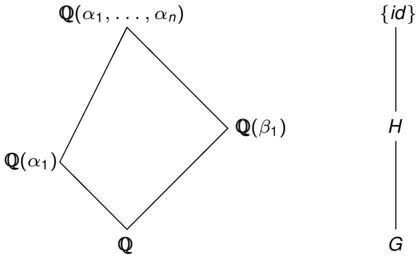
Computation of Galois groups: degree 24 and beyond

Jürgen Klüners

Universität Paderborn

8th Nov 2014 / Barcelona

Fundamental theorem of Galois theory



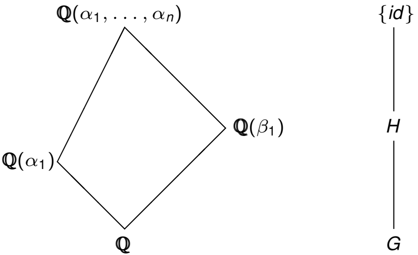
G permutes the zeros $\alpha_1, \ldots, \alpha_n$ of the minimal polyn. $f \in \mathbb{Z}[x]$ of α_1. The subgroups of G are in bijection to the subfields of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

Discriminant criterion

$G \leq A_n \Leftrightarrow \mathbf{Disc}(f) = (-1)^{n(n-1)/2} Res(f, f') = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$ is a square in $\mathbb{Z}$.

For $n = 3$ this gives an algorithm to compute Galois groups.

Fundamental theorem of Galois theory



G permutes the zeros $\alpha_1, \ldots, \alpha_n$ of the minimal polyn. $f \in \mathbb{Z}[x]$ of α_1. The subgroups of G are in bijection to the subfields of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

Discriminant criterion

$G \leq A_n \Leftrightarrow \mathbf{Disc}(f) = (-1)^{n(n-1)/2} Res(f, f') = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$ is a square in $\mathbb{Z}$.

For $n = 3$ this gives an algorithm to compute Galois groups.

Setup

- $f \in \mathbb{Z}[x]$ square-free and monic
- α_1, ..., α_n roots of f in some field
- $\Gamma := \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ splitting field.

Then $G := \mathbf{Aut}(\Gamma/\mathbb{Q}) \leq \mathbf{Sym}(\alpha_1, \ldots, \alpha_n) \sim S_n$ is the Galois group of f.

Aim: To compute G.

Note: This is more than just the structure of G or the conjugacy class.

The trivial approach

The construction of a generic splitting field by factorization already gives a method for the computation of G.

Example

$f(x) := x^4 - 3 \in \mathbb{Q}[x]$ (which is irreducible).
Set $K_1 := \mathbb{Q}(\alpha) \sim \mathbb{Q}[x] / \langle f \rangle$.
Factorize f over K_1:

$$f = (x - \alpha)(x + \alpha)(x^2 + \alpha^2).$$

Next we adjoin a root of the quadratic factor to K_1:

$$K_2 := K_1(\beta) \sim K_1[x] / \langle x^2 + \alpha^2 \rangle.$$

Example $f(x) = x^4 - 3$ continued

- $K_1 := \mathbb{Q}(\alpha) \sim {}^{\mathbb{Q}[x]}/_{\langle f \rangle}$.
- $f = (x-\alpha)(x+\alpha)(x^2+\alpha^2) \in K_1[x]$.
- $K_2 := K_1(\beta) \sim {}^{K_1[x]}/_{\langle x^2+\alpha^2 \rangle}$.

Splitting field $\Gamma := K_2 = \mathbb{Q}(\alpha, \beta)$

- $G = \mathbf{Gal}(f)$ acts on the roots $\pm\alpha$ and $\pm\beta$.
- There is only one transitive subgroup of the $\mathbf{Sym}(4)$ of order 8.
- The Galois group is D_4 - but how does G act on the roots?

As a permutation group acting on $(\alpha, -\alpha, \beta, -\beta)$, G is generated by $(1,4,2,3)$ and $(1,4)(2,3)$.

Some useful tools

Cycle types, $p \nmid \mathbf{Disc}(f)$ and $f \equiv f_1 \dots f_r \bmod p\mathbb{Z}[x]$

$\mathbf{Gal}(f)$ contains an element π of cycle type $(\deg(f_1), \dots, \deg(f_r))$.
If $G = S_n$, this will determined quickly.

Resolvent method

$$F_m(x) := \prod_{1 \le i_1 < \dots < i_m \le n} (x - (\alpha_{i_1} + \dots \alpha_{i_m})) \in \mathbb{Z}[x].$$

If F_m is squarefree, then the degrees of the factors of F_m coincide with the orbit lengths of the operation of $\mathbf{Gal}(f)$ on the m-sets.
F_m can be computed only using the coefficients.

Problems and history

Problems

- Computation of splitting fields not efficient, degree might be $n!$.
- How to represent the roots of f?

- Stauduhar, 1973, method up to degree 7
- Geyer, 1992, degree 9
- Eichenlaub, Olivier, 1995, degree 11
- Geißler, 1997, degree 12, still using complex approximations
- Geißler, Klüners, 2000, degree 15, p-adic approximations
- Geißler, 2003, degree 23

Current implementation in Magma

- Fieker-Klüners (without degree restriction)
- Many improvements by Stephan Elsenhans (invariant theory)

The Stauduhar algorithm I

Goal

Compute Galois groups including the action on the roots

Idea

- Situation: $\mathbf{Gal}(f) \leq G$, $H < G$ maximal subgroup.
- Decide, if $\mathbf{Gal}(f) \leq H^\tau$ for a $\tau \in G//H$.
- Here $\mathbf{Gal}(f)$ operates on the zeros $\alpha_1, \ldots, \alpha_n$ of f.

The Stauduhar algorithm II

Relative invariants

- Compute $F \in \mathbb{Z}[x_1, \ldots, x_n]$ with $F^H = F$ (pointwise) and $F^g \neq F \;\; \forall g \in G \setminus H$.
- Therefore $R_{G,H,F}(x) := \prod_{\tau \in G//H} (x - F^\tau(\alpha_1, \ldots, \alpha_n)) \in \mathbb{Z}[x]$.

Main theorem of Stauduhar

Let $R_{G,H,F}(x)$ be squarefree. Then: $\mathbf{Gal}(f) \leq H^\tau \Leftrightarrow F^\tau(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}$.

Problems

- Computation of maximal subgroups up to conjugation.
- Computation of the invariants.
  - Special invariants (wreath products, index-2-subgroups,...).
  - Example: $\prod_{1 \le j < k \le n} (x_j - x_k)$ is A_n–invariant, S_n–relative.
- Representation of $\alpha_1, \ldots, \alpha_n$ ($\to$ p-adic approximation).
- Decide, if $F^\tau(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}$.
- Large indices $(G : H)$ (e.g. 40 million in degree 13,14).
  - Computation of $G//H$.
  - p-adic Precision is $p^k > (2M)^{(G:H)}$.

Maximal transitive subgroups of S_n and A_n:

$$
\begin{array}{ll}
(S_{14} : 14T_{61}) = 1716 & (A_{14} : 14T_{59}^{+}) = 3432 \\
(S_{14} : 14T_{57}) = 135135 & (A_{14} : 14T_{55}^{+}) = 270270 \\
(S_{14} : 14T_{39}) = 39916800 & (A_{14} : 14T_{30}^{+}) = 39916800
\end{array}
$$

$$
\begin{array}{ll}
(S_{15} : 15T_{102}) = 126126 & (A_{15} : 15T_{99}^{+}) = 126126 \\
(S_{15} : 15T_{93}) = 1401400 & (A_{15} : 15T_{89}^{+}) = 1401400 \\
 & (A_{15} : 15T_{72}^{+}) = 32432400
\end{array}
$$

$$
(S_{23} : AGL(1,23)) = 21! \qquad (A_{23} : M_{23}) = 1.267.136.462.592.000
$$

Unramified p-adic extensions

- $p \nmid \mathbf{Disc}(f)$ be a prime number and $f \equiv f_1 \dots f_r \bmod p$.
- Then $\bar{\alpha}_1, \dots, \bar{\alpha}_n \subset \mathbb{F}_q$, where $q = p^\ell$ with $\ell := \mathbf{lcm}(\deg(f_i))$.
- $E := \{\sum_{i=n_0}^{\infty} a_i p^i \mid a_i \in \mathbb{F}_q, n_0 \in \mathbb{Z}\}$ is unramified over $\mathbb{Q}_p$.
- $\alpha_1, \dots, \alpha_n \in o_E = \{\sum_{i=0}^{\infty} a_i p^i \mid a_i \in \mathbb{F}_q\}$.
- Modulo p^k-approximation: $\sum_{i=0}^{k-1} a_i p^i$.

When is a number in $\mathbb{Z}$?

Is 1.000000000001 an integer or perhaps 1.57?

$$\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Q}_p \subset E.$$

Easy observation: If $a \in E \setminus \mathbb{Q}_p \Rightarrow a \notin \mathbb{Z}$.

How can we prove that *a* in $\mathbb{Z}$?

Situation

$a \equiv b \bmod p^k$ and $|a| < M$ with $p^k > 2M$.

Choose b in the symmetric residue system $\{\frac{-(p^k-1)}{2}, \ldots, \frac{p^k-1}{2}\}$.
If $|b| > M$, then $a \notin \mathbb{Z}$.
If $|b| < M$, then: If $a \in \mathbb{Z}$, then $a = b$.

Theorem

Let $M \in \mathbb{R}$ with $|F^\sigma(\alpha_1, \ldots, \alpha_n)| < M \ \forall \sigma \in S_n$ and $p^k > (2M)^{(G:H)}$.
Then:
$F^\tau(\alpha_1, \ldots, \alpha_n) = b \in \mathbb{Z} \Leftrightarrow F^\tau(\alpha_1, \ldots, \alpha_n) \equiv b \bmod p^k$ *and* $|b| < M$.

Proof: $R_{G,H,F}(x) = \prod_{\sigma \in G//H} (x - F^\sigma(\alpha_1, \ldots, \alpha_n)) \in \mathbb{Z}[x]$

Then: $R_{G,H,F}(b) < (2M)^{(G:H)}$ and congruent to 0 modulo p^k.

Subfields

Computation of non-trivial subfields

- gives smaller starting group.
- avoids the most difficult descents.

- Subfield computation is more efficient than Galois group computation. (My PhD-thesis, 1997).
- Generating Subfields, joint with Mark van Hoeij, Andrew Novocin, 2013
  - Determination of all generating subfields in polynomial time (degree, size of coefficient).
  - All subfields can be computed as intersections of those (linear in the number of subfields).

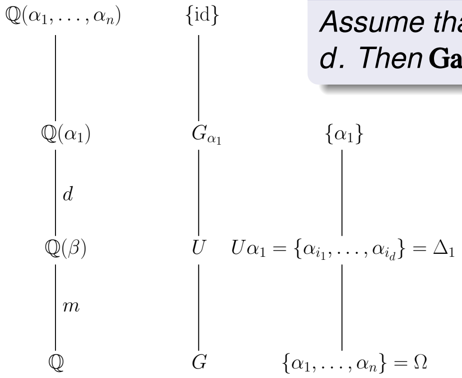
Subfields II

Definition

$\emptyset \neq \Delta \subseteq \Omega$ is called block, if $\Delta^\tau \cap \Delta \in \{\emptyset, \Delta\}$ for all $\tau \in G$. The orbit $\Delta_1, \ldots, \Delta_m$ of a block Δ_1 of G is called block system.

Lemma

Assume that $\mathbf{Gal}(f)$ *has a block* Δ *of size* d*. Then* $\mathbf{Gal}(f) \leq S_d \wr S_m$*.*



Subfields III

Problem

Determine $\mathbf{Gal}(f) \leq S_d \wr S_m$ *including the operation on* $\alpha_1, \ldots, \alpha_n$.

- $L = \mathbb{Q}(\beta_1) \subset K = \mathbb{Q}(\alpha_1)$, $\beta_1, \ldots, \beta_m$ are the conjugates of β_1.
- Then there exists an $h \in \mathbb{Q}[x]$ with $h(\alpha_1) = \beta_1$.
- We have $h(\alpha_i) = \beta_j$ for some $1 \leq j \leq m$.

Lemma

Define $\Delta_j := \{\alpha_i \mid h(\alpha_i) = \beta_j\}$. *Then* $\Delta_1, \ldots, \Delta_m$ *is a block system corresponding to* L

This gives an improvement for imprimitive Galois groups.

Short Cosets

- Is $\mathbf{Gal}(f) \leq \tau H \tau^{-1}$ for a $\tau \in G//H$?
- Known: Frobenius–automorphism $\sigma \in \mathbf{Gal}(f)$ with $\sigma(\alpha_i) \equiv \alpha_i^p \bmod p$.
- $(G//H)_\sigma := \{\tau \in G//H \mid \sigma \in \tau H \tau^{-1}\}$,
- explicitly computable via centralizer computation.

Example

$H := 14T_{39}^{+} \cong PGL_2(13) < G := S_{14}$, $(G : H) = 39.916.800$
Choose $\sigma = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14)$ and we get $|(G//H)_\sigma| = 1$.

Short Cosets

- Is $\mathbf{Gal}(f) \leq \tau H \tau^{-1}$ for a $\tau \in G//H$?
- Known: Frobenius–automorphism $\sigma \in \mathbf{Gal}(f)$ with $\sigma(\alpha_i) \equiv \alpha_i^p \bmod p$.
- $(G//H)_\sigma := \{\tau \in G//H \mid \sigma \in \tau H \tau^{-1}\}$,
- explicitly computable via centralizer computation.

Example

$H := 14T_{39}^{+} \cong PGL_2(13) < G := S_{14}$, $(G : H) = 39.916.800$
Choose $\sigma = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14)$ and we get $|(G//H)_\sigma| = 1$.

Proof for primitive groups

Problem:

$(G//H)_\sigma$ small, but $p^k > (2M)^{(G:H)}$.

Choose k with $p^k > (2M)^{10}$ and get $\mathbf{Gal}(f) = H$ incl. operation on the roots with "high probability".

Is $\mathbf{Gal}(f) = H$ or $\mathbf{Gal}(f) = G$, where $H < G$?

Proof using the so-called resolvent method.

Implementations
with Claus Fieker

Algorithm for arbitrary degree

- over $\mathbb{Q}$, other ground fields in Magma
- Complete rewrite!
- Maximal subgroups computation (already done in magma)
- Computation of special invariants (big improvements by Stephan Elsenhans)
- Choosing a good prime (for short cosets, cheap arithmetic)
- Many new problems in higher degrees, e.g. computation of conjugacy classes for big 2-groups

Easy special invariants

Reminder: $H < G$ maximal subgroup, compute:
$F \in \mathbb{Z}[x_1, \ldots, x_n]$ with $F^H = F$ (pointwise) and $F^g \neq F \;\; \forall g \in G \setminus H$.
If $(G : H)$ is small, then G and H are almost equal.

Other interpretation:

F is a primitive element of the field extension

$$\mathbb{Q}(x_1, \ldots, x_n)^H / \mathbb{Q}(x_1, \ldots, x_n)^G.$$

Example

H_1, H_2, H_3 3 subgroups of G of index 2 with $H_1 \cap H_2 \subset H_3 \subset G$. Then determine invariant F_3 from invariants F_1 and F_2. If done properly, we get $F_3 = F_1 F_2$.

Special invariants – block systems

H has a block system

$$\{x_1, \ldots, x_d\}, \ldots, \{x_{(m-1)d+1}, \ldots, x_n\},$$

which is not a block system of G. Then:

$$F(x_1, \ldots, x_n) := (x_1 + \ldots + x_d) \cdots (x_{(m-1)d+1} + \ldots + x_n)$$

$\Delta_1, \ldots, \Delta_m$ block system of H and G

- $\tilde{H}, \tilde{G}$ operation of H and G, resp., on $\Delta_1, \ldots, \Delta_m$. If $\tilde{H} \subsetneq \tilde{G}$, then $\tilde{H}$–invariant $\tilde{F}(y_1, \ldots, y_m)$ produces $F(x_1, \ldots, x_n) = \tilde{F}(x_1 + \ldots + x_d, \ldots, x_{(m-1)d+1} + \ldots + x_n)$.
- Analogue reduction, if operation within the blocks is different.

Special invariants – wreath products

Idea:

Generalization of diskriminant criterion: $G = S_d \wr S_m$.

$$d_k := \prod_{1 \le i < j \le d} (x_{i,k} - x_{j,k}), \quad (1 \le k \le m)$$

s_k elementary symmetric polynomials $(1 \le k \le m)$

$$D := \prod_{1 \le i < j \le m} (y_i - y_j) \text{ with } y_j = x_{(j-1)d+1} + \ldots + x_{jd}$$

Lemma

$S_d \wr S_m$ has at least 3 subgroups of index 2, i.e. the stabilizers of $s_m(d_1, \ldots, d_m)$, $D(y_1, \ldots, y_m)$ (this is $S_d \wr A_m$) and $D(y_1, \ldots, y_m)s_m(d_1, \ldots, d_m)$.

Explicit realization of transitive groups of small degree

Problem $G \leq S_n$ transitive

Compute a polynomial $f \in \mathbb{Q}[x]$ with $\mathbf{Gal}(f) = G$.

Known theoretical results

- All solvable groups over $\mathbb{Q}$ are realizable.
- S_n, A_n, all abelian groups, and all sporadic groups with the possible exception M_{23} are realizable over $\mathbb{Q}$ (and regularly over $\mathbb{Q}(t)$).

degree	6	7	8	9	10	11	12	13	14	15	16
number	16	7	50	34	45	8	301	9	63	104	1954

Results

Theorem (Klüners-Malle, 2000)

All transitive groups up to degree 15 are regularly realizable over $\mathbb{Q}(t)$.

Theorem (Klüners-Malle, 2000)

For all transitive groups up to degree 15 we have computed a polynomial $f \in \mathbb{Q}[x]$ *with* $\mathbf{Gal}(f) = G$.

Current progress: Realizations over $\mathbb{Q}$

Degree 16: All 1954 groups are realized
Degree 17: All groups, but $L_2(16):2$ ($L_2(16)$ realized by J. Bosman)
Degree 18: All 983 groups over $\mathbb{Q}$ realized
Degree 19-23 All groups (except M_{23}) over $\mathbb{Q}$ realized

Construction methods

- Compute polynomial for other permutation representation or quotient
- Direct products $A \times H$
- Wreath products $A \wr H$ of A with H.
- Split extensions $A \rtimes H$ with abelian kernel A
- Subdirect products (fiber products)
- Rigidity and generalizations
- Class field theory

Inductive reduction to smaller groups

Let $H \lneq G$ and $A \trianglelefteq G$ abelian with $G = AH$. Then G is quotient of $A \rtimes H$.

Irreducible non semiabelian groups

Definition

G is called **irreducible non semiabelian**, if there exists no subgroup $H \lneq G$ with $G = AH$ for some abelian normal subgroup A.

12 (23)	13 (3)	14 (13)	15 (18)
12T57	$L_3(3)$	$L_2(13)$	$A_5 \wr 3$
12T124		14T33	15T94
$PGL_2(11)$		$PGL_2(13)$	15T95
12T287		$L_2(7) \wr 2$	$A_5 \wr S_3$
M_{12}		$A_7 \wr 2$	15T97-100
$A_6 \wr 2$		14T59	$S_5 \wr 3$
12T297-298		14T60	$S_5 \wr S_3$
$S_6 \wr 2$		$S_7 \wr 2$	

Critical groups are: 12T57, 12T124, 14T33.

Goal of the database

- Compute a polynomial for each group and signature (Number of real roots).
- Determine the number field with the smallest discriminant for each entry.

Web address

galoisdb.math.uni-paderborn.de

Theorem (Serre)

All groups are realizable over $\mathbb{Q}$ $\Leftrightarrow$ all groups are realizable over $\mathbb{Q}$ with a totally real polynomial.

Missing entries

Missing entries for signatures up to degree 15

totally real: $L_2(13)$, $PGL_2(13)$

- Emmanuel Hallouine: $9T_{27} = L_2(8)$ and $9T_{32} = P\Gamma L_2(8)$.
- Joachim König: $PGL_2(11)$, $L_3(3)$.

Minima

- Complete up to degree 7
- Complete in degree 8 for imprimitive groups
- Many groups in degrees 9-12

Primitive groups in degree 8

G	$r_1 =$ 0	2	4	6	8
$8T_{25}$	594823321	—	—	—	9745585291264
$8T_{36}$	1817487424	—	—	—	6423507767296
$8T_{37}$	$\leq$ 37822859361	—	—	—	$\leq$ 235163942523136
$8T_{43}$	$\leq$ 418195493	≥ -1997331875	—	—	$\leq$ 312349488740352
$8T_{48}$	$\leq$ 32684089	—	$\leq$ 293471161	—	$\leq$ 81366421504
$8T_{49}$	$\leq$ 20912329	—	$\leq$ 144889369	—	$\leq$ 46664208361
$8T_{50}$	$\leq$ 1282789	≥ -4296211	$\leq$ 15908237	≥ -65106259	483345053

Thank you very much for your attention

Web address of the database

galoisdb.math.uni-paderborn.de